Density perturbations with relativistic thermodynamics

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We investigate cosmological density perturbations in a covariant and gauge-invariant formalism, incorporating relativistic causal thermodynamics to give a self-consistent description. The gradient of density inhomogeneities splits covariantly into a scalar part, equivalent to the usual density perturbations, a rotational vector part that is determined by the vorticity, and a tensor part that describes the shape. We give the evolution equations for these parts in the general dissipative case. Causal thermodynamics gives evolution equations for viscous stress and heat flux, which are coupled to the density perturbation equation and to the entropy and temperature perturbation equations. We give the full coupled system in the general dissipative case, and simplify the system in certain cases. A companion paper uses the general formalism to analyze damping of density perturbations before last scattering.

I. INTRODUCTION

The analysis of density perturbations in cosmological fluids is well established, particularly using Bardeen's gauge invariant formalism [1,2]. This formalism is inherently linear (i.e., it starts from the background and perturbs away from it) and non-local. An alternative approach, developed by Ellis and Bruni [3], is covariant (and therefore local) and readily incorporates nonlinear effects (since it starts from the real spacetime, not the background). We will use this covariant and gauge-invariant formalism, in which the variables have a clear physical and geometric interpretation. Furthermore, the covariant approach is directly compatible with causal relativistic thermodynamics, as developed by Israel and Stewart [4].

Although dissipative terms representing viscosity and heat conduction have been formally incorporated into the equations in both approaches [1,5], most applications of the theory are restricted to the non-dissipative case – and even in this case, relativistic thermodynamics is usually not applied to analyze the behavior of the fluid self-consistently. This is is not a problem when studying the evolution of large-scale perturbations, which are unaffected by local physics – although the generation of these perturbations, their initial evolution before leaving the Hubble radius, and their final evolution after re-entering the Hubble radius, are governed by local physics. For small-scale perturbations, within the Hubble radius, a self-consistent analysis requires the application of thermodynamics.[1]

Here and in a companion paper [8], we develop and apply a covariant and gauge-invariant analysis of density perturbations that self-consistently incorporates relativistic causal thermodynamics. The general evolution equations governing density inhomogeneities are considered in Sec. II. Inhomogeneities are covariantly characterized by a scalar part, which represents the usual density perturbations, a vector part, which we show is determined by the vorticity, and a tensor part, which determines the shape of gravitational clustering. New evolution equations are derived for the vector and tensor parts, as well as for perturbations in the number density, entropy and temperature. We use the Gibbs equation to incorporate the temperature and entropy self-consistently, and we covariantly characterize different types of perturbation. In Sec. III, the viscous stress and heat flux that appear in the perturbation evolution equations are subject to thermodynamical transport equations, which then form a coupled system with the perturbation evolution equations. We define appropriate dissipative scalars to obtain a closed system of dynamical equations. The equations

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[1]We are considering here the case of hydrodynamics. Dissipative effects on the microwave background have been self-consistently analyzed via numerical integration of the Boltzmann equation (see, e.g., [6]). A covariant and gauge-invariant approach to the Boltzmann equation is developed in [7].

are simplified in the particular cases of entropy perturbations (non-dissipative), and when only one form of dissipation is present.

The Israel-Stewart transport equations are under reasonable conditions causal and stable [9], and thus provide a consistent relativistic description of local physical effects on small-scale perturbations. The thermodynamics of Eckart (and a similar alternative due to Landau and Lifshitz) is more established in the literature. However, in this theory the transport equations reduce from evolution equations to algebraic constraints on viscosity and heat flux, and as a result, the theory is non-causal (dissipative effects propagate at super-luminal speeds), and all its equilibrium states are unstable [9]. It can be argued [10] that these pathologies only occur outside the hydrodynamic regime. But firstly, the stability problem persists in all situations, and secondly, it seems preferable to employ a theory with built-in causality and stability. Furthermore, the causal theory can deal with transient and short-wavelength effects, which are important in many cosmological and astrophysical situations (see, e.g., [11–14]).

Applications to dissipative situations are treated in a companion paper [8], where we analyze viscous damping of density perturbations before last scattering. This generalizes the results of Weinberg [15], who used non-causal Eckart thermodynamics.

II. COVARIANT APPROACH TO DISSIPATIVE PERTURBATIONS

The covariant and gauge-invariant analysis of density perturbations is fully discussed in [3,5]. Here we present only the main points that are necessary for our purposes, before going further by deriving new evolution equations and incorporating causal thermodynamics. Our notation and conventions follow [5,16,17], with some changes (see [18]).

Given a covariantly defined fluid four-velocity u^a (see the further discussion below), then $h_{ab} = g_{ab} + u_a u_b$ projects into the local rest spaces of comoving observers, where g_{ab} is the spacetime metric. The covariant $1+3$ splitting of the Bianchi identities and the Ricci identity for u^a, incorporating Einstein's field equations as an algebraic definition of the Ricci tensor, $R_{ab} = T_{ab} - \frac{1}{2}Tg_{ab}$, are the fundamental equations in the covariant perturbation approach. These equations may be written as propagation and constraint equations for covariant scalars, spatial vectors ($V_a = h_a{}^b V_b$) and spatial 2-tensors which are symmetric and trace-free, i.e. which satisfy

$$S_{ab} = S_{\langle ab\rangle} \equiv h_a{}^c h_b{}^d S_{(cd)} - \tfrac{1}{3}h_{cd}S^{cd}h_{ab} \,.$$

Any spatial 2-tensor has the covariant irreducible decomposition

$$S_{ab} = \tfrac{1}{3}Sh_{ab} + S_{\langle ab\rangle} + \varepsilon_{abc}S^c \,,$$

where $S \equiv h^{ab}S_{ab}$ is the spatial trace and $S_a = \frac{1}{2}\varepsilon_{abc}S^{bc}$ is the spatial dual to the skew part. Here $\varepsilon_{abc} = \eta_{abcd}u^d$ is the spatial permutation tensor defined by projecting the spacetime permutation tensor η_{abcd}. The covariant derivative ∇_a splits into a covariant time derivative $\dot{A}_{a\cdots} = u^b\nabla_b A_{a\cdots}$, and a covariant spatial derivative $\mathrm{D}_b A_{a\cdots} = h_b{}^d h_a{}^c \cdots \nabla_d A_{c\cdots}$. (Note that $\mathrm{D}_c h_{ab} = 0 = \mathrm{D}_d\varepsilon_{abc}$.) Then the covariant spatial divergence and curl are defined by [16]

$$\operatorname{div} V = \mathrm{D}^a V_a \,, \quad \operatorname{curl} V_a = \varepsilon_{abc}\mathrm{D}^b V^c \,,$$
$$(\operatorname{div} S)_a = \mathrm{D}^b S_{ab} \,, \quad \operatorname{curl} S_{ab} = \varepsilon_{cd(a}\mathrm{D}^c S_{b)}{}^d \,.$$

The fluid kinematics are described by the scalar $\theta = \mathrm{D}^a u_a$ (expansion), the spatial vectors $\dot{u}_a$ (four-acceleration) and $\omega_a = -\frac{1}{2}\operatorname{curl} u_a$ (vorticity), and the tensor $\sigma_{ab} = \mathrm{D}_{\langle a}u_{b\rangle}$ (shear). The locally free gravitational field is described by the electric and magnetic parts of the Weyl tensor, $E_{ab} = C_{acbd}u^c u^d = E_{\langle ab\rangle}$ and $H_{ab} = \frac{1}{2}\varepsilon_{acd}C^{cd}{}_{be}u^e = H_{\langle ab\rangle}$. The fluid dynamics are given by the energy density ρ, the pressure p, and the dissipative quantities B (bulk viscous stress), Q_a (heat flux, $Q_a u^a = 0$) and $\pi_{ab} = \pi_{\langle ab\rangle}$ (shear viscous stress). These arise in the energy-momentum tensor

$$T_{ab} = \rho u_a u_b + (p+B)h_{ab} + 2q_{(a}u_{b)} + \pi_{ab} \,, \tag{1}$$

where [4]

$$q_a = Q_a + \left(\frac{\rho+p}{n}\right) j_a \tag{2}$$

is the total energy flux relative to u^a, with n the particle number density and j_a the particle flux ($j_a u^a = 0$). The latter are combined in the particle four-flow vector

$$N^a = nu^a + j^a \,. \tag{3}$$

In a self-consistent thermo-hydrodynamic description, we need to introduce also the temperature T and specific entropy s per particle, defined in, or near to, equilibrium via the Gibbs equation

$$Tds = d\Big(\frac{\rho}{n}\Big) + p\, d\Big(\frac{1}{n}\Big)\,. \tag{4}$$

The hydrodynamic tensors T_{ab} and N^a define two natural four-velocities – the particle (or Eckart) four-velocity $u^a_{\rm p}$, for which $j_a = 0 \Leftrightarrow Q_a = q_a$, and the energy (or Landau-Lifshitz) four-velocity $u^a_{\rm e}$, for which $q_a = 0 \Leftrightarrow Q_a = -(\rho+p)j_a/n$. These four-velocities coincide in equilibrium, and differ by a small angle near to equilibrium.[2] The four-velocity u^a may be chosen to be close to $u^a_{\rm p}$ and $u^a_{\rm e}$. Any small change in u^a produces second order changes (negligible in the linear regime) in ρ, p, n, T, and s [4]. These scalars therefore coincide (to first order) with the corresponding scalars for a local equilibrium state. The bulk and shear viscous stresses B and π_{ab} are also invariant to first order under a small change in u^a. Both q^a and j^a undergo first-order changes, but the heat flux vector Q^a is invariant to first order. From Eq. (1), we see that

$$u^a = u^a_{\rm e} - \frac{1}{(\rho+p)} q^a\,. \tag{5}$$

For the isotropic and homogeneous Friedmann-Lemaitre-Robertson-Walker (FLRW) universes

$$\begin{aligned}
&{\rm D}_a\theta = {\rm D}_a\rho = {\rm D}_a p = {\rm D}_a B = {\rm D}_a n = {\rm D}_a T = {\rm D}_a s = 0\,,\\
&\dot u_a = \omega_a = q_a = j_a = 0\,,\\
&\sigma_{ab} = E_{ab} = H_{ab} = \pi_{ab} = 0\,.
\end{aligned}$$

In covariant perturbation theory, a universe with small anisotropy and inhomogeneity is characterized by these quantities being small, and one neglects terms which are nonlinear in them. Since these quantities vanish in the background, they are gauge-invariant [3]. Note that FLRW models can admit scalar dissipation, in the form of a bulk viscous stress B (see, e.g., [12–14,19]), reflecting the fact that expanding fluids in general cannot maintain equilibrium [4]. However, we shall follow the standard approach in irreversible thermodynamics of assuming an equilibrium background state, so that $B = 0$ in the background.[3] For convenience, the linearized Bianchi and Ricci equations that underlie the covariant gauge-invariant theory are given in Appendix A (using the above notation and definitions, introduced in [16], which considerably simplify the original equations). Appendix A also contains useful differential identities. Note that in the background

$$\theta = 3H\,,\ \rho = 3H^2(1+K)\,,\ \dot H = -\tfrac{1}{2}H^2\left[3(1+w) + (1+3w)K\right]\,,\ B = 0 = \dot s\,, \tag{6}$$

where $H = \dot a/a$ is the Hubble rate, a is the cosmic scale factor, $K = 0, \pm(aH)^{-2}$ is the dimensionless spatial curvature index, and $w = p/\rho$.

Linearization of the number conservation equation $\nabla_a N^a = 0$ gives

$$\dot n + \theta n = -{\rm D}^a j_a\,. \tag{7}$$

Using the energy conservation equation (A1) and the number conservation equation (7), together with Eq. (2), the Gibbs equation (4) implies that

$$nT\dot s = -3HB - {\rm D}^a Q_a\,. \tag{8}$$

The contribution of shear viscous stress to entropy generation is via a nonlinear term $\sigma^{ab}\pi_{ab}$, so that *in an almost-FLRW universe, the shear viscous stress does not contribute to* $\dot s$. Thus non-dissipative perturbations are not adequately characterized by $\dot s = 0$. We need to specify that $B = Q_a = \pi_{ab} = 0$.

Scalar perturbations are covariantly and gauge-invariantly characterized by the spatial gradients of scalars. Density inhomogeneities are described by the comoving fractional density gradient [3]

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[2] It has recently been argued [20] that only the energy frame is suitable for the description of irreversible thermodynamics.

[3] The consistency condition that this assumption imposes through the transport equation (40) for B, is that the bulk viscosity ζ should be much less than $\rho\theta^{-1}$.

$$\delta_a = \frac{a\mathrm{D}_a\rho}{\rho}\,. \tag{9}$$

We define also the comoving expansion gradient [3], and the dimensionless fractional number density gradient (not considered in [3,5]), normalized pressure gradient, and normalized entropy gradient (see [18]) by

$$\theta_a = a\mathrm{D}_a\theta\,,\ \nu_a = \frac{a\mathrm{D}_a n}{n}\,,\ p_a = \frac{a\mathrm{D}_a p}{\rho}\,,\ e_a = \frac{anT\mathrm{D}_a s}{\rho}\,. \tag{10}$$

Using the fact that $p = p(\rho, s)$, and the Gibbs equation (4), we find

$$p_a = c_{\rm s}^2\delta_a + re_a\,, \tag{11}$$

$$e_a = \delta_a - (1+w)\nu_a\,, \tag{12}$$

where the dimensionless quantities

$$c_{\rm s}^2 = \left(\frac{\partial p}{\partial\rho}\right)_s,\quad r = \frac{1}{nT}\left(\frac{\partial p}{\partial s}\right)_\rho, \tag{13}$$

are respectively the adiabatic speed of sound and a non-barotropic index. Note that in equations (11) and (12), these quantities and w are evaluated in the background.[4] In the background

$$c_{\rm s}^2 = \frac{\dot p}{\dot\rho}\,,\ \dot w = -3H(1+w)(c_{\rm s}^2 - w)\,, \tag{14}$$

where we have used $\dot s = 0$ and the energy conservation equation (A1).

A covariant thermodynamic classification of scalar perturbations is as follows. Perturbations are *non-dissipative* if $B = Q_a = \pi_{ab} = 0$, and then in particular $\dot s = 0$, so that the specific entropy is constant along fluid flow-lines. If the specific entropy is the same universal constant along all flow-lines, i.e. if $e_a = 0$ in addition to $\dot s = 0$, then the perturbations are *isentropic*, often (misleadingly) called 'adiabatic'. For isentropic perturbations, equations (11) and (12) show that the number density perturbations and pressure perturbations are algebraically determined by the energy density perturbations: $\nu_a = \delta_a/(1+w)$, $p_a = c_{\rm s}^2\delta_a$. The case of dissipative perturbations with $e_a = 0$, so that the specific entropy varies, but only along the fluid flow, will be called *dissipative perturbations without entropy perturbations*. The integrability condition $\dot e_a = 0$, implies, via the gradient of the entropy evolution equation (8), that

$$3H\mathrm{D}_a B + \mathrm{D}_a\left(\mathrm{D}^b Q_b\right) = 0\,, \tag{15}$$

which is very restrictive, except in the case where only shear viscous stress is present. In general, dissipative perturbations will also involve entropy perturbations.

The evolution of the temperature is clearly affected by the nature of the perturbations. In order to determine how this works, we use ρ and s as the independent thermodynamic variables in the Gibbs equation (4). The integrability condition $\partial^2 n/\partial\rho\partial s = \partial^2 n/\partial s\partial\rho$ then gives

$$(\rho + p)\left(\frac{\partial T}{\partial\rho}\right)_s = \left(c_{\rm s}^2 + r\right)T\,,$$

where we used

$$\left(\frac{\partial n}{\partial s}\right)_\rho = -\frac{n^2 T}{\rho + p}\,,$$

which follows from the Gibbs equation. Using the identity

$$\dot T = \left(\frac{\partial T}{\partial\rho}\right)_s\dot\rho + \left(\frac{\partial T}{\partial s}\right)_\rho\dot s\,,$$

[4]If B is nonzero in the background, then the background speed of sound acquires an additional dissipative contribution $c_{\rm b}$, where (see [21]) $c_{\rm b}^{-2} = \beta_0(\rho + p)$, and β_0 arises in Eq. (40).

together with the energy conservation equation (A1) and the entropy evolution equation (8), the above equations lead to the *temperature evolution equation*

$$\frac{\dot{T}}{T} = -\left(c_{\rm s}^2 + r\right)\theta - \frac{\left(c_{\rm s}^2 + r\right)}{\rho(1+w)}\left[3HB + {\rm D}^a q_a\right] - \frac{1}{nT^2}\left(\frac{\partial T}{\partial s}\right)_\rho \left[3HB + {\rm D}^a Q_a\right] . \tag{16}$$

This equation reproduces the standard cooling rates for perfect fluids in the nonrelativistic and ultra-relativistic cases.[5] In the general case, the source terms on the right hand side show the role of non-barotropic and dissipative effects. Note that the last term vanishes if the temperature is barotropic, i.e., if $T = T(\rho)$. Bulk viscous perturbations counteract the cooling due to expansion, shear viscous perturbations do not affect the cooling rate (to first order), and the effect of heat flux depends on the sign of the divergence. For non-dissipative perturbations, the sign of the non-barotropic index r determines whether cooling is enhanced or retarded.[6]

We can also derive new evolution equations for the number density perturbations, entropy perturbations, and temperature perturbations. The comoving gradient of the number conservation equation (7), together with the momentum conservation equation (A2) and the identity (A15), gives

$$\dot{\nu}_a + 3rH\nu_a = -\theta_a + 3(1+w)^{-1}\left(c_{\rm s}^2 + r\right)H\delta_a + \frac{a}{\rho(1+w)}\left[3H\left({\rm D}_a B + \dot{q}_a + 4Hq_a + {\rm D}^b\pi_{ab}\right) + {\rm D}_a{\rm D}^b(Q_b - q_b)\right] . \tag{17}$$

The comoving gradient of the entropy evolution equation (8), together with the energy conservation equation (A1), the temperature evolution equation (16), and the identity (A15), gives the *evolution equation for entropy perturbations:*

$$\dot{e} + 3H\left(c_{\rm s}^2 - w + r\right)e = -\frac{a^2}{\rho}\left[3H{\rm D}^2 B + {\rm D}^2\left({\rm D}^a Q_a\right)\right] , \tag{18}$$

where we have defined the scalar entropy perturbation

$$e = a{\rm D}^a e_a = \frac{a^2 nT}{\rho}{\rm D}^2 s\,. \tag{19}$$

Eq. (18) shows that *for non-dissipative or shear viscous perturbations, entropy perturbations decay with expansion unless* $c_{\rm s}^2 - w + r \leq 0$. Defining the comoving fractional temperature gradient by

$$T_a = \frac{a{\rm D}_a T}{T}\,, \tag{20}$$

we find from the evolution equation (16) and the identity (A15), that the *evolution equation of (covariant and gauge-invariant) temperature perturbations* is given by

$$\dot{T}_a = -3\left(c_{\rm s}^2 + r\right)Ha\dot{u}_a - \left(c_{\rm s}^2 + r\right)\theta_a - 3Ha{\rm D}_a\left(c_{\rm s}^2 + r\right) - \frac{a\left(c_{\rm s}^2 + r\right)}{\rho(1+w)}\left[3H{\rm D}_a B + {\rm D}_a\left({\rm D}^b q_b\right)\right] - \frac{a}{nT^2}\left(\frac{\partial T}{\partial s}\right)_\rho\left[3H{\rm D}_a B + {\rm D}_a\left({\rm D}^b Q_b\right)\right] . \tag{21}$$

Now δ_a contains more information than just the scalar density perturbations, since at each point, δ_a picks out the direction of maximal inhomogeneity. The irreducible parts of the comoving gradient of δ_a then describe completely and covariantly the variation in density inhomogeneities:

$$a{\rm D}_b\delta_a = \tfrac{1}{3}\delta\, h_{ab} + \xi_{ab} + \varepsilon_{abc}W^c\,, \tag{22}$$

[5] A similar equation is given in [21], but in the particle frame only, and without separating out the non-barotropic r terms.

[6] Using the Gibbs equation (4), we can show that $r = c_{\rm s}^2[\rho(1+w)\alpha - nc_{\rm p}]/nc_{\rm p}$, where $\alpha = n(\partial n^{-1}/\partial T)_p \geq 0$ is the dilatation coefficient, and $c_{\rm p} = T(\partial s/\partial T)_p \geq 0$ is the specific heat at constant pressure.

where the scalar part $\delta \equiv a\mathrm{D}^a\delta_a = (a\mathrm{D})^2\rho/\rho$ corresponds to the usual gauge-invariant density perturbation scalar ε_{m} [1], the vector part $W_a = -\frac{1}{2}a\,\mathrm{curl}\,\delta_a$ describes the rotational properties of inhomogeneous clustering, and the tensor part $\xi_{ab} = a\mathrm{D}_{\langle a}\delta_{b\rangle}$ describes the volume-true distortion of inhomogeneous clustering. (These quantities were introduced in [5], but only the scalar δ was discussed.) These irreducible parts encode respectively the total scalar, vector and tensor contributions to density inhomogeneities.

It is difficult to see how a rotation independent of the vorticity could arise, and indeed we can show that W_a is always proportional to the vorticity vector:

$$W_a = -3a^2H(1+w)\omega_a\,. \tag{23}$$

This follows from the identity (A13) and the energy conservation equation (A1). Thus rotation in clustering matter is inherited entirely (in the linear regime) from cosmic rotation: *the vector part of density inhomogeneities is determined completely in direction by the cosmic vorticity.* The expansion and pressure index affect the magnitude of the vector part. In particular, it follows that $W_a = 0$ if the background is non-expanding or De Sitter ($w = -1$).

The vorticity propagation equation (A4) leads to the new *evolution equation for the vector part of density inhomogeneities:*

$$\dot{W}_a + \tfrac{1}{2}H\left[3(1-w)+(1+3w)K\right]W_a = -\left(\frac{3a^2H}{2\rho}\right)\mathrm{curl}\left[\dot{q}_a + 4Hq_a + \mathrm{D}^b\pi_{ab}\right], \tag{24}$$

where we have used equations (6) and (14), and the momentum conservation equation (A2) allowed us to evaluate $\mathrm{curl}\,\dot{u}_a$, together with the identity (A13). Unsurprisingly, Eq. (24) shows that the scalar dissipative quantity B does not influence the evolution of the vector part of density inhomogeneities.[7] In the energy frame, heat flux also has no direct influence on W_a. Eq. (24) shows that *for non-dissipative or only shear viscous perturbations, W_a decays with expansion unless* $3(1-w)+(1+3w)K \leq 0$. For ordinary hydrodynamic matter, with $0 \leq w \leq \frac{1}{3}$, this inequality is never satisfied if the spatial curvature is non-negative.

To derive evolution equations for the tensor and scalar parts, we take the comoving spatial gradient of the energy conservation equation (A1) and the Raychaudhuri equation (A3), using the momentum conservation equation (A2), the identities (A14)–(A16), and Eq. (23):

$$\dot{\delta}_a = 3wH\delta_a - (1+w)\theta_a + \frac{3aH}{\rho}\left[\dot{q}_a + 4Hq_a + \mathrm{D}^b\pi_{ab}\right] - \frac{a}{\rho}\mathrm{D}_a\mathrm{D}^bq_b\,, \tag{25}$$

$$\begin{aligned}(1+w)\dot{\theta}_a = {}& -2H(1+w)\theta_a - \tfrac{3}{2}H^2\left[1+w+\left(1+w+\tfrac{2}{3}c_{\mathrm{s}}^2\right)K\right]\delta_a - c_{\mathrm{s}}^2\mathrm{D}^2\delta_a \\ & -r\left(KH^2+\mathrm{D}^2\right)e_a - \frac{a}{\rho}\left(KH^2+\mathrm{D}^2\right)\mathrm{D}_aB \\ & +\frac{3aH^2}{2\rho}\left[3(1+w)+(1+3w)K\right]\left[\dot{q}_a + 4Hq_a + \mathrm{D}^b\pi_{ab}\right] \\ & -\frac{a}{\rho}\mathrm{D}_a\mathrm{D}^b\left[\dot{q}_b + 4Hq_b + \mathrm{D}^c\pi_{bc}\right] + \frac{2}{a}c_{\mathrm{s}}^2\,\mathrm{curl}\,W_a\,.\end{aligned} \tag{26}$$

Eq. (25) can be shown to be in agreement with equation (61) of [5], while Eq. (26) generalizes equation (62) of [5] by including bulk viscous effects.

We can now decouple the equations:

$$\begin{aligned}&\ddot{\delta}_a + H\left(2-6w+3c_{\mathrm{s}}^2\right)\dot{\delta}_a - \tfrac{3}{2}H^2\left[1+8w-3w^2-6c_{\mathrm{s}}^2+\left(1-3w^2+\tfrac{2}{3}c_{\mathrm{s}}^2\right)K\right]\delta_a \\ &= c_{\mathrm{s}}^2\mathrm{D}^2\delta_a - \frac{2}{a}c_{\mathrm{s}}^2\,\mathrm{curl}\,W_a + r\left(KH^2+\mathrm{D}^2\right)e_a + \frac{a}{\rho}\left(KH^2+\mathrm{D}^2\right)\mathrm{D}_aB \\ &+3\frac{aH}{\rho}\left\{\ddot{q}_a + H\left[7-3w+3c_{\mathrm{s}}^2-(1+3w)K\right]\dot{q}_a\right. \\ &\left.+6H^2\left[1-3w+2c_{\mathrm{s}}^2-(1+3w)K\right]q_a - c_{\mathrm{s}}^2\mathrm{D}_a\mathrm{D}^bq_b\right\} \\ &+\frac{a}{\rho}\left\{3H\mathrm{D}^b\dot{\pi}_{ab} + 3H^2\left[2-3w+3c_{\mathrm{s}}^2-(1+3w)K\right]\mathrm{D}^b\pi_{ab} + \mathrm{D}_a\mathrm{D}^b\mathrm{D}^c\pi_{bc}\right\}\,,\end{aligned} \tag{27}$$

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[7] Although the gradient of B occurs in the transport equation (41) for Q_a, and therefore occurs on the right hand side of (24) in the particle frame ($q_a = Q_a$), the curl of this gradient is negligible by the identity (A13), since B vanishes in the background.

where we have used equations (6), (9), (11), and (A14)–(A16). The comoving gradient of the evolution equation (27) determines evolution equations for the scalar, vector and tensor parts of density inhomogeneities, incorporating all dissipative and entropy effects. We have already derived the vector evolution equation (24). Taking the comoving divergence of equation (27), and using identities (A14)–(A16) and (A20), gives the *evolution equation for scalar density perturbations*

$$\begin{aligned}&\ddot{\delta}+H\left(2-6w+3c_{\rm s}^2\right)\dot{\delta}-\tfrac{3}{2}H^2\left[1+8w-3w^2-6c_{\rm s}^2+\left(1-3w^2+2c_{\rm s}^2\right)K\right]\delta-c_{\rm s}^2{\rm D}^2\delta\\&\quad=\mathsf{S}[e]+\mathsf{S}[B]+\mathsf{S}[q]+\mathsf{S}[\pi]\,,\end{aligned}\tag{28}$$

where the source terms arising respectively from entropy perturbations, bulk viscous stress, energy flux, and shear viscous stress are:

$$\mathsf{S}[e]=r\left(3KH^2+{\rm D}^2\right)e\,,\tag{29}$$

$$\mathsf{S}[B]=\left(3KH^2+{\rm D}^2\right)\mathcal{B}\,,\tag{30}$$

$$\begin{aligned}\mathsf{S}[q]=3aH\,\big\{&\ddot{q}+H\left[1-9w+c_{\rm s}^2-(1+3w)K\right]\dot{q}\\&-\tfrac{3}{2}H^2\left[1+8w-9w^2-8c_{\rm s}^2+(1-9w^2)K\right]q-c_{\rm s}^2{\rm D}^2q\big\}\,,\end{aligned}\tag{31}$$

$$\mathsf{S}[\pi]=3H\dot{\mathcal{S}}-3H^2\left[1+6w-3c_{\rm s}^2+(1+3w)K\right]\mathcal{S}+{\rm D}^2\mathcal{S}\,,\tag{32}$$

and we have defined the dimensionless perturbation scalars

$$\mathcal{B}=\frac{a^2{\rm D}^2B}{\rho}\,,\ q=\frac{a{\rm D}^aq_a}{\rho}\,,\ \mathcal{S}=\frac{a^2{\rm D}^a{\rm D}^b\pi_{ab}}{\rho}\,.\tag{33}$$

Eq. (28) generalizes equation (74) of [5] to include bulk viscous effects, and is presented we believe in a more transparent form, which makes clear the physical meaning of each term.

The new *evolution equation for the tensor part of density inhomogeneities* follows from the trace-free symmetric part of the comoving gradient of Eq. (27), on using the identities (A14)–(A16):

$$\begin{aligned}&\ddot{\xi}_{ab}+\left(2-6w+3c_{\rm s}^2\right)H\dot{\xi}_{ab}-\tfrac{3}{2}H^2\left[1+8w-3w^2-6c_{\rm s}^2+\left(1-3w^2+\tfrac{2}{3}c_{\rm s}^2\right)K\right]\xi_{ab}\\&-c_{\rm s}^2{\rm D}_{\langle a}{\rm D}_{b\rangle}\delta=\mathsf{S}[e]_{ab}+\mathsf{S}[B]_{ab}+\mathsf{S}[q]_{ab}+\mathsf{S}[\pi]_{ab}\,.\end{aligned}\tag{34}$$

The source terms are given by

$$\mathsf{S}[e]_{ab}=3rKaH^2{\rm D}_{\langle a}e_{b\rangle}+r{\rm D}_{\langle a}{\rm D}_{b\rangle}e\,,\tag{35}$$

$$\mathsf{S}[B]_{ab}=3K\frac{a^2H^2}{\rho}{\rm D}_{\langle a}{\rm D}_{b\rangle}B+{\rm D}_{\langle a}{\rm D}_{b\rangle}\mathcal{B}\,,\tag{36}$$

$$\begin{aligned}\mathsf{S}[q]_{ab}=3\frac{a^2H}{\rho}\big\{&{\rm D}_{\langle a}\ddot{q}_{b\rangle}+H\left[7-3w+3c_{\rm s}^2-(1+3w)K\right]{\rm D}_{\langle a}\dot{q}_{b\rangle}\\&+6H^2\left[1-3w+2c_{\rm s}^2-(1+3w)K\right]{\rm D}_{\langle a}q_{b\rangle}-c_{\rm s}^2{\rm D}_{\langle a}{\rm D}_{b\rangle}{\rm D}^cq_c\big\}\,,\end{aligned}\tag{37}$$

$$\begin{aligned}\mathsf{S}[\pi]_{ab}=\frac{a^2}{\rho}\Big\{&3H{\rm D}_{\langle a}{\rm D}^c\dot{\pi}_{b\rangle c}+3H^2\left[2-3w+3c_{\rm s}^2-(1+3w)K\right]{\rm D}_{\langle a}{\rm D}^c\pi_{b\rangle c}\\&+{\rm D}_{\langle a}{\rm D}_{b\rangle}{\rm D}^c{\rm D}^d\pi_{cd}\Big\}\,.\end{aligned}\tag{38}$$

Comparison of equations (28) and (34) shows that in the simplest case of isentropic perturbations, the density distortion tensor ξ_{ab} obeys the same equation as the scalar δ, so that

$$\xi_{ab}=A_{ab}\,\delta\,,\quad \dot{A}_{ab}=0\,.\tag{39}$$

The presence of entropy or dissipative perturbations breaks the simple relation (39), and the evolution of the shape of density inhomogeneities is not directly determined by the scalar density perturbation.

III. CAUSAL TRANSPORT EQUATIONS

We are now ready to introduce the evolution equations obeyed by the dissipative quantities in the causal thermodynamics of Israel and Stewart [4]. This theory is based on a covariant treatment of the second law of thermodynamics

and the conservation equations, and its transport equations are confirmed by relativistic kinetic theory (via the relativistic generalization of the Grad approximation), which also provides explicit expressions for the various thermodynamic parameters in the case of a dilute gas. The theory thus has a firm physical foundation. Furthermore, as pointed out earlier, dissipative signals propagate below the speed of light and the equilibrium states are stable, within the regime of validity of the theory. Thus the causal and stable thermodynamics of Israel and Stewart is a consistent relativistic thermodynamics which supercedes the non-causal and unstable theories first put forward by Eckart and Landau & Lifshitz.

The predictions of the causal theory agree with those of the pathological theories in quasi-stationary situations. But when high-frequency/ short-wavelength effects are important, in the transient regime, the pathological theories are inapplicable. Thus these theories cannot cover the full range of behaviour of a relativistic fluid near equilibrium. Moreover, these theories cannot even constitute part of a consistent theoretical thermo-hydrodynamics because they are intrinsically not relativistic theories, given their pathologies. Thus our approach is to employ the causal thermodynamics to construct a self-consistent theory of cosmological density perturbations in the general case. In particular applications, where it can be argued that the non-causal theories will give reasonable results, we can then specialize the general equations appropriately. This is done in the companion paper [8].

The full form of the transport equations, encompassing situations where the background equilibrium state is accelerating and rotating, and including terms which were neglected in the original theory and restored by Hiscock and Lindblom [9], is given in Appendix B for convenience. Since we are dealing with cosmological perturbations, the background is non-rotating and non-accelerating, and spatial gradients of thermodynamic coefficients give rise to nonlinear terms. There are also linear terms, containing time derivatives of thermodynamic coefficients, which were restored by [9]. We will follow the arguments of [21,14] which show that under many reasonable conditions, these terms may be neglected in comparison with the other terms in the transport equations.[8]

With these simplifications, equations (B1)–(B3) reduce to the *causal transport equations*

$$B = -\zeta\left[\theta + \beta_0\dot{B} - \alpha_0 \mathrm{D}^a Q_a\right] , \tag{40}$$

$$Q_a = -\kappa\left[\mathrm{D}_a T + T\dot{u}_a + T\beta_1\dot{Q}_a - T\alpha_0\mathrm{D}_a B - T\alpha_1\mathrm{D}^b\pi_{ab}\right] , \tag{41}$$

$$\pi_{ab} = -2\eta\left[\sigma_{ab} + \beta_2\dot{\pi}_{ab} - \alpha_1\mathrm{D}_{\langle a}Q_{b\rangle}\right] . \tag{42}$$

The coefficients ζ, κ, and η of bulk viscosity, thermal conductivity, and shear viscosity, appear also in the non-causal (and the non-relativistic) theories. The coefficients β_I define characteristic relaxational time-scales

$$\tau_0 = \zeta\beta_0\,,\ \ \tau_1 = \kappa T\beta_1\,,\ \ \tau_2 = 2\eta\beta_2\,,$$

which are often taken to be of the order of the mean collision time, but which are determined by collisional integrals in kinetic theory [4]. The non-causal theories are characterized by $\beta_I = 0$. Intuitively, this corresponds to instantaneous relaxation to equilibrium when the dissipative 'force' is switched off. The coefficients α_I, which also vanish in the non-causal case, arise from a coupling of viscous stress and heat flux (see Appendix B). They may also be found from kinetic theory in the case of a dilute gas. These transport equations hold in the energy and particle frames, with suitable simple changes in some of the thermodynamic coefficients (see Appendix B).

The transport equations (40)–(42) are coupled to the evolution equations for density inhomogeneities – the scalar equation (28), the vector equation (24), and the tensor equation (34). They are also coupled to the evolution equations (17), (18) and (21) for number density, entropy, and temperature perturbations. In all of these evolution equations, except the entropy evolution equation (18), the energy flux vector q_a occurs. Considerable simplification is thus achieved by choosing the energy frame ($q_a = 0$), which is consistent with arguments in favour of that frame [20,22].

In general, the coupling amongst the transport and evolution equations is highly complicated, although the coupled system can always be cast into a form suitable for numerical integration. Even in the non-dissipative case, when the transport equations fall away, the evolution equations themselves are coupled. In the simplest case of isentropic perturbations ($B = Q_a = \pi_{ab} = 0$, $e = 0$), the scalar equation (28) reduces to a wave equation only in δ (with undamped phase speed c_{s}^2, as expected). In principle, the solution of this equation, and the solution W_a of Eq. (24), may be used to express Eq. (34) as an equation only in ξ_{ab}.

For non-dissipative entropy perturbations, decoupling δ leads to a third-order equation. First we apply the operator $3KH^2 + \mathrm{D}^2$ to Eq. (18), using identity (A17) and

[8]Note however that it is not always reasonable to neglect these terms - see [12,13] for examples.

$$\dot{K} = H(1+3w)\,(1+K)\,K\,,$$

to get

$$\left[\left(3KH^2+\mathrm{D}^2\right)e\right]^{\cdot} + H\left(2-3w+3c_{\rm s}^2+3r\right)\left[\left(3KH^2+\mathrm{D}^2\right)e\right] = 0\,.$$

Then we use Eq. (28) in this latter equation, together with identity (A17) and Eq. (6), to obtain the *decoupled density perturbation evolution equation for non-dissipative entropy perturbations*

$$\begin{aligned}&\ddot{\delta}^{\cdot} + \left[\left(4-9w+6c_{\rm s}^2+3r\right)H - (\ln r)^{\cdot}\right]\ddot{\delta} - \tfrac{1}{2}AH\dot{\delta} - \tfrac{3}{2}BH^2\delta \\ &\quad = c_{\rm s}^2\mathrm{D}^2\dot{\delta} + \left[\left\{3\left(c_{\rm s}^2-w+r\right)H - (\ln r)^{\cdot}\right\}c_{\rm s}^2 + \left(c_{\rm s}^2\right)^{\cdot}\right]\mathrm{D}^2\delta\,,\end{aligned} \tag{43}$$

where A and B are complicated functions of $w, c_{\rm s}^2, r, H$, and K. In the case of a flat background $K=0$, we have

$$\begin{aligned}A &= \left(1+84w-27w^2-69c_{\rm s}^2+27wc_{\rm s}^2-12r+36wr-18rc_{\rm s}^2-18c_{\rm s}^4\right)H \\ &\quad - 6\left(c_{\rm s}^2\right)^{\cdot} + 2\left(2-6w-3c_{\rm s}^2\right)(\ln r)^{\cdot}\,, \\ B &= \left(-1+10w-39w^2+24rw-15c_{\rm s}^2+54wc_{\rm s}^2+9w^2c_{\rm s}^2+3r-9w^2r-18rc_{\rm s}^2-18c_{\rm s}^4\right)H \\ &\quad - 6\left(c_{\rm s}^2\right)^{\cdot} - \left(1+8w-3w^2-6c_{\rm s}^2\right)(\ln r)^{\cdot}\,.\end{aligned}$$

A. The coupled system in general

The complexity of Eq. (43) indicates the difficulty of decoupling the equations for dissipative perturbations. In general, the fact that the transport equations are first-order in time derivatives shows that any decoupling will produce at least one higher time derivative in the evolution equations. In the non-causal limit ($\beta_I = 0$), when the derivative terms drop out of the transport equations, this does not hold, and the order of the equations is the same as in the non-dissipative case.

The transport equations (40)–(42) contain further couplings amongst the dissipative quantities and couplings to the density and entropy and temperature perturbations. These further couplings are revealed when we take comoving spatial gradients, and use the following expressions:

$$\frac{a\mathrm{D}_a\zeta}{\rho} = \left(\frac{\partial\zeta}{\partial\rho}\right)_s\delta_a + \frac{1}{nT}\left(\frac{\partial\zeta}{\partial s}\right)_\rho e_a\,,$$

which follows from equation (10);

$$a\mathrm{D}^a\theta_a = \frac{1}{1+w}\left[3wH\delta - \dot{\delta} + 3H\mathcal{S}\right]\,,$$

which follows from Eq. (25), using the energy frame;

$$a\mathrm{D}^a\dot{u}_a = -\frac{1}{a(1+w)}\left[c_{\rm s}^2\delta + re + \mathcal{B} + \mathcal{S}\right]\,,$$

which follows from the momentum conservation equation (A2);

$$a^2\mathrm{D}^a\mathrm{D}^b\sigma_{ab} = \tfrac{2}{3}a\mathrm{D}^a\theta_a\,,$$

which follows from the constraint equation (A6) and the identity (A20); and

$$\mathrm{D}^a\mathrm{D}^b\mathrm{D}_{\langle a}Q_{b\rangle} = \left(\rho - 3H^2\right)\mathrm{D}^aQ_a + \tfrac{2}{3}\mathrm{D}^2\left(\mathrm{D}^aQ_a\right)\,,$$

which follows from identity (A18). Then operating on the transport equations (40)–(42) with, respectively, $(a^2/\rho)\mathrm{D}^2$, $(a/\rho)\mathrm{D}^a$, and $(a^2/\rho)\mathrm{D}^a\mathrm{D}^b$, we get the new *causal transport equations for the scalar dissipative quantities*

$$
\begin{aligned}
&\tau_0\dot{\mathcal{B}} + [1-3(1+w)\tau_0 H]\,\mathcal{B} = (\zeta\alpha_0 a)\mathrm{D}^2\mathcal{Q} + \left[\frac{3\zeta H}{\rho(1+w)}\right]\mathcal{S} \\
&\quad + \frac{\zeta}{\rho(1+w)}\left[\dot{\delta} - 3H\left\{w + (1+w)\frac{\rho}{\zeta}\left(\frac{\partial\zeta}{\partial\rho}\right)_s\right\}\delta\right] - \left[\frac{3H}{nT}\left(\frac{\partial\zeta}{\partial s}\right)_\rho\right]e\,, \qquad (44)\\
&\tau_1\dot{\mathcal{Q}} + [1-3(1+w)\tau_1 H]\,\mathcal{Q} = -\left(\frac{\kappa T}{a\rho}\right)\mathcal{T} \\
&\quad + \frac{\kappa T}{a\rho(1+w)}\left[\{\alpha_0\rho(1+w)-1\}\,\mathcal{B} + \{\alpha_1\rho(1+w)-1\}\,\mathcal{S}\right] + \frac{\kappa T}{a\rho(1+w)}\left[c_\mathrm{s}^2\delta + re\right]\,, \qquad (45)\\
&\tau_2\dot{\mathcal{S}} + \left[1 - H\left\{3(1+w)\tau_2 - \frac{4\eta}{1+w}\right\}\right]\mathcal{S} = \\
&\quad \tfrac{2}{3}\eta\alpha_1 a\left(\mathrm{D}^2\mathcal{Q} + 3H^2K\mathcal{Q}\right) + \frac{4\eta}{3(1+w)}\left[\dot{\delta} - 3wH\delta\right]\,. \qquad (46)
\end{aligned}
$$

We have defined the scalars for heat flux and temperature perturbations

$$
\mathcal{Q} = \frac{a\mathrm{D}^aQ_a}{\rho}\,,\ \mathcal{T} = a\mathrm{D}^aT_a\,. \qquad (47)
$$

The comoving spatial divergence of equation (21) gives the new *evolution equation for scalar temperature perturbations:*

$$
\begin{aligned}
\dot{\mathcal{T}} = &\left(\frac{c_\mathrm{s}^2 + r}{1+w}\right)\left[\dot{\delta} + 3H(c_\mathrm{s}^2 - w)\delta + 3Hre\right] - 3Ha^2\mathrm{D}^2(c_\mathrm{s}^2 + r) \\
&- \frac{\rho}{nT^2}\left(\frac{\partial T}{\partial s}\right)_\rho\left[3H\mathcal{B} + a\mathrm{D}^2\mathcal{Q}\right] + 3H\left(\frac{c_\mathrm{s}^2 + r}{1+w}\right)\mathcal{S}\,. \qquad (48)
\end{aligned}
$$

In summary, *the coupled system that governs scalar dissipative perturbations in the general case* is given by: the density perturbation equation (28), the entropy perturbation equation (18), which we rewrite as

$$
\dot{e} + 3H\left(c_\mathrm{s}^2 - w + r\right)e = -3H\mathcal{B} - a\mathrm{D}^2\mathcal{Q}\,, \qquad (49)
$$

and the equations (44)–(46) and (48).

The number density perturbations do not occur in the coupled system. Once δ, $\mathcal{B}$, and $\mathcal{Q}$ are determined from the coupled system, the scalar number density perturbations, defined by $\nu = a\mathrm{D}^a\nu_a$, are found from the comoving divergence of equation (17). In the energy frame, this gives

$$
\dot{\nu} + 3rH\nu = (1+w)^{-1}\left[\dot{\delta} + 3\left(c_\mathrm{s}^2 - w + r\right)H\delta + 3H\mathcal{B} + a^2\mathrm{D}^2\mathcal{Q}\right]\,. \qquad (50)
$$

The new evolution equation (50) shows how bulk viscous stress and heat flux govern the deviation of number density perturbations from energy density perturbations. Note that shear viscous stress does not directly affect the number density perturbations.

For specific applications, we present below the simplified coupled system that arises in two special cases when only one form of dissipation is present.

B. Bulk viscous stress only

The coupled system can be reduced to a pair of coupled equations in δ (second-order in time) and e (second-order in time). In principle these may be decoupled. For a flat background, the equations are:

$$
\begin{aligned}
&\ddot{\delta} + H\left(2 - 6w + 3c_\mathrm{s}^2\right)\dot{\delta} - \tfrac{3}{2}H^2\left(1 + 8w - 3w^2 - 6c_\mathrm{s}^2\right)\delta \\
&\quad = c_\mathrm{s}^2\mathrm{D}^2\delta + \left(w - c_\mathrm{s}^2\right)\mathrm{D}^2 e - \frac{1}{3H}\mathrm{D}^2\dot{e}\,, \qquad (51)
\end{aligned}
$$

and

$$\tau_0\ddot{e} + \left[1 - \tfrac{3}{2}\left(1+3w-2c_{\rm s}^2-2r\right)\tau_0 H\right]\dot{e}$$
$$- 3H\left[w - c_{\rm s}^2 - r + 3(1+w)r\tau_0 H - \tau_0\left(c_{\rm s}^2+r\right)^{\cdot} + \frac{3H}{nT}\left(\frac{\partial\zeta}{\partial s}\right)_\rho\right] e$$
$$= -\left[\frac{\zeta}{H(1+w)}\right]\dot{\delta} + \frac{3}{(1+w)}\left[w\zeta + \rho(1+w)\left(\frac{\partial\zeta}{\partial\rho}\right)_s\right]\delta\,. \tag{52}$$

Once these equations are solved for δ and e, the other scalar quantities may be determined. Note that by the consistency condition (15), the entropy perturbations cannot be zero unless $\mathcal{B}$ itself vanishes.

C. Shear viscous stress only

In the absence of entropy perturbations, the consistency condition (15) is identically satisfied if only shear viscous stress is present, and the system reduces to the pair of coupled equations:

$$\ddot{\delta} + H\left(2-6w+3c_{\rm s}^2\right)\dot{\delta} - \tfrac{3}{2}H^2\left[1+8w-3w^2-6c_{\rm s}^2 + \left(1-3w^2+2c_{\rm s}^2\right)K\right]\delta$$
$$= c_{\rm s}^2{\rm D}^2\delta + 3H\dot{\mathcal{S}} - 3H^2\left[1+6w-3c_{\rm s}^2+(1+3w)K\right]\mathcal{S} + {\rm D}^2\mathcal{S}\,, \tag{53}$$
$$\tau_2\dot{\mathcal{S}} + \left[1 - H\left\{3(1+w)\tau_2 - \frac{4\eta}{1+w}\right\}\right]\mathcal{S}$$
$$= \frac{4\eta}{3(1+w)}\left[\dot{\delta} - 3wH\delta\right]\,. \tag{54}$$

Finally, we point out an interesting feature of the case when only shear viscous stress is present (with or without entropy perturbations). The vector part W_a of density inhomogeneities satisfies the decoupled wave equation

$$\tau_2\ddot{W}_a + \left[1 + \tfrac{3}{2}\left\{1-w+(1+3w)K\right\}\tau_2 H\right]\dot{W}_a$$
$$+ \tfrac{1}{4}H\left[6(1-w) - (1+w)\left(3-w-6c_{\rm s}^2\right)\tau_2 H + \frac{24\eta}{\rho H}\right.$$
$$+ \left\{2(1+3w) + \left[2(1+6w+3w^2) - 6(1+w)c_{\rm s}^2 + 3(1+3w)^2K\right]\tau_2 H\right.$$
$$\left.\left. - \frac{8\eta}{\rho H}\left(\frac{1-3w}{1+w}\right)\right\}K\right]W_a = \left[\frac{\eta}{\rho(1+w)}\right]{\rm D}^2 W_a\,. \tag{55}$$

The term ${\rm D}^b\sigma_{ab}$ that arises from the transport equation (42) is eliminated via the constraint equation (A6). We used the identity (A22) and the constraint equation (A7) to evaluate $\mathrm{curl\,curl}\,\omega_a$. The undamped phase speed v is clearly given by

$$v^2 = \frac{\eta}{\tau_2(\rho+p)}\,.$$

It follows from the analysis of [9] that $v \leq 1$, as expected from causality requirements.

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APPENDIX A: COVARIANT PROPAGATION AND CONSTRAINT EQUATIONS

The Ricci identity for u^a and the Bianchi identities (incorporating the field equations via the Ricci tensor), may be covariantly split into propagation and constraint equations (see [5]). In [17], these equations are given in our streamlined notation in the exact nonlinear case, for a perfect fluid. For an almost-FLRW universe, with imperfect fluid, the linearized form of the equations follows. (In the non-dissipative case, all right hand sides are zero.)

$$\dot\rho + (\rho+p)\theta = -3HB - \mathrm{D}^a q_a \,, \tag{A1}$$
$$(\rho+p)\dot u_a + \mathrm{D}_a p = -\mathrm{D}_a B - \dot q_a - 4Hq_a - \mathrm{D}^b\pi_{ab} \,, \tag{A2}$$
$$\dot\theta + \tfrac{1}{3}\theta^2 - \mathrm{D}^a\dot u_a + \tfrac{1}{2}(\rho+3p) = -\tfrac{3}{2}B \,, \tag{A3}$$
$$\dot\omega_a + 2H\omega_a + \tfrac{1}{2}\mathrm{curl}\,\dot u_a = 0 \,, \tag{A4}$$
$$\dot\sigma_{ab} + 2H\sigma_{ab} - \mathrm{D}_{\langle a}\dot u_{b\rangle} + E_{ab} = \tfrac{1}{2}\pi_{ab} \,, \tag{A5}$$
$$\tfrac{2}{3}\mathrm{D}_a\theta - \mathrm{D}^b\sigma_{ab} + \mathrm{curl}\,\omega_a = q_a \,, \tag{A6}$$
$$\mathrm{D}^a\omega_a = 0 \,, \tag{A7}$$
$$H_{ab} - \mathrm{curl}\,\sigma_{ab} - \mathrm{D}_{\langle a}\omega_{b\rangle} = 0 \,, \tag{A8}$$
$$\dot E_{ab} + 3HE_{ab} - \mathrm{curl}\,H_{ab} + \tfrac{1}{2}(\rho+p)\sigma_{ab} = -\tfrac{1}{2}\dot\pi_{ab} - \tfrac{1}{2}H\pi_{ab} - \tfrac{1}{2}\mathrm{D}_{\langle a}q_{b\rangle} \,, \tag{A9}$$
$$\dot H_{ab} + 3HH_{ab} + \mathrm{curl}\,E_{ab} = \tfrac{1}{2}\mathrm{curl}\,\pi_{ab} \,, \tag{A10}$$
$$\mathrm{D}^b E_{ab} - \tfrac{1}{3}\mathrm{D}_a\rho = -Hq_a - \tfrac{1}{2}\mathrm{D}^b\pi_{ab} \,, \tag{A11}$$
$$\mathrm{D}^b H_{ab} - (\rho+p)\omega_a = -\tfrac{1}{2}\mathrm{curl}\,q_a \,. \tag{A12}$$

Some useful differential identities are [3,16]

$$\mathrm{curl}\,\mathrm{D}_a f = -2\dot f\omega_a \,, \tag{A13}$$
$$\mathrm{D}^2\left(\mathrm{D}_a f\right) = \mathrm{D}_a\left(\mathrm{D}^2 f\right) + \tfrac{2}{3}\left(\rho - 3H^2\right)\mathrm{D}_a f + 2\dot f\,\mathrm{curl}\,\omega_a \,, \tag{A14}$$
$$\left(\mathrm{D}_a f\right)^{\cdot} = \mathrm{D}_a\dot f - H\mathrm{D}_a f + \dot f\dot u_a \,, \tag{A15}$$
$$\left(a\mathrm{D}_a A_{b\cdots}\right)^{\cdot} = a\mathrm{D}_a\dot A_{b\cdots} \,, \tag{A16}$$
$$\left(\mathrm{D}^2 f\right)^{\cdot} = \mathrm{D}^2\dot f - 2H\mathrm{D}^2 f + \dot f\mathrm{D}^a\dot u_a \,, \tag{A17}$$
$$\mathrm{D}_{[a}\mathrm{D}_{b]}V_c = \left(H^2 - \tfrac{1}{3}\rho\right)V_{[a}h_{b]c} \,, \tag{A18}$$
$$\mathrm{D}_{[a}\mathrm{D}_{b]}S^{cd} = 2\left(H^2 - \tfrac{1}{3}\rho\right)S_{[a}{}^{(c}h_{b]}{}^{d)} \,, \tag{A19}$$
$$\mathrm{D}^a\mathrm{curl}\,V_a = 0 \tag{A20}$$
$$\mathrm{D}^b\mathrm{curl}\,S_{ab} = \tfrac{1}{2}\mathrm{curl}\,\left(\mathrm{D}^b S_{ab}\right) \,, \tag{A21}$$
$$\mathrm{curl}\,\mathrm{curl}\,V_a = \mathrm{D}_a\left(\mathrm{D}^b V_b\right) - \mathrm{D}^2 V_a + 2\left(\tfrac{1}{3}\rho - H^2\right)V_a \,, \tag{A22}$$
$$\mathrm{curl}\,\mathrm{curl}\,S_{ab} = \tfrac{3}{2}\mathrm{D}_{\langle a}\mathrm{D}^c S_{b\rangle c} - \mathrm{D}^2 S_{ab} + \left(\rho - 3H^2\right)S_{ab} \,, \tag{A23}$$

where the vectors and tensors vanish in the background, $S_{ab} = S_{\langle ab\rangle}$, and all the identities except (A13) are linearized.

APPENDIX B: FULL CAUSAL TRANSPORT EQUATIONS

For completeness and convenience, we amalgamate the results of [4] [equations (7.1)][9] and [9] [equations (21)–(23)][10] in our notation, and present the causal transport equations for viscous stress and heat flux in the general (near equilibrium) case, covering both cosmological and other (e.g., astrophysical) scenarios.

$$\begin{aligned} B = &-\zeta\left[\theta+\beta_0\dot{B}-\alpha_0\mathrm{D}^aQ_a\right] \\ &+\zeta\left\{a_0'\dot{u}^aQ_a+\gamma_0TQ^a\mathrm{D}_a\left(\frac{\alpha_0}{T}\right)-\tfrac{1}{2}B\left[\beta_0\theta+T\left(\frac{\beta_0}{T}\right)^{\cdot}\right]\right\}\,, \end{aligned} \tag{B1}$$

$$\begin{aligned} Q_a = &-\kappa T\left[\frac{1}{T}\mathrm{D}_aT+\dot{u}_a+\beta_1\dot{Q}_{\langle a\rangle}-\alpha_0\mathrm{D}_aB-\alpha_1\mathrm{D}^b\pi_{ab}\right] \\ &+\kappa T\left\{a_0B\dot{u}_a+a_1\dot{u}^b\pi_{ab}+\beta_1\varepsilon_{abc}Q^b\omega^c\right\} \\ &+\kappa T\left\{(1-\gamma_0)TB\mathrm{D}_a\left(\frac{\alpha_0}{T}\right)+(1-\gamma_1)T\pi_a{}^b\mathrm{D}_b\left(\frac{\alpha_1}{T}\right)-\tfrac{1}{2}Q_a\left[\beta_1\theta+T\left(\frac{\beta_1}{T}\right)^{\cdot}\right]\right\}\,, \end{aligned} \tag{B2}$$

$$\begin{aligned} \pi_{ab} = &-2\eta\left[\sigma_{ab}+\beta_2\dot{\pi}_{\langle ab\rangle}-\alpha_1\mathrm{D}_{\langle a}Q_{b\rangle}\right] \\ &+2\eta\left\{a_1'\dot{u}_{\langle a}Q_{b\rangle}+2\beta_2\varepsilon_{cd(a}\pi_{b)}{}^c\omega^d\right\} \\ &+2\eta\left\{\gamma_1TQ_{\langle a}\mathrm{D}_{b\rangle}\left(\frac{\alpha_1}{T}\right)-\tfrac{1}{2}\pi_{ab}\left[\beta_2\theta+T\left(\frac{\beta_2}{T}\right)^{\cdot}\right]\right\}\,, \end{aligned} \tag{B3}$$

where $\dot{Q}_{\langle a\rangle}\equiv h_a{}^b\dot{Q}_b$. The coefficients ζ, κ, and η are, respectively, the bulk viscosity, thermal conductivity, and shear viscosity. The relaxation coefficients β_0, β_1, and β_2 are crucial to the causal behavior of the theory. The coefficients α_0 and α_1 arise from a coupling between viscous stress and heat flux, as reflected in the entropy four-current [4]

$$\begin{aligned} S^a = &\,sN^a+\frac{1}{T}Q^a+\frac{\alpha_0}{T}BQ^a+\frac{\alpha_1}{T}\pi_{ab}Q^b \\ &-\frac{1}{2T}\left(\beta_0B^2+\beta_1Q_bQ^b+\beta_2\pi_{bc}\pi^{bc}\right)u^a \\ &+\frac{1}{2T(\rho+p)}\left(q^bq_bu^a+2\pi^{ab}q_b\right)\,. \end{aligned} \tag{B4}$$

The coefficients a_0, a_0', a_1, and a_1' mediate the coupling of acceleration and vorticity to viscous stress and heat flux, while γ_0 and γ_1 appear due to a coupling of the spatial gradients of α_I to viscous stress and heat flux. There are simple relations between the unprimed and primed a_I [4].

A change from energy to particle frame results in a re-definition of various thermodynamic coefficients, but the transport equations maintain the same form. A partial comparison is given in [4] (p. 350), but the heat flux equation in the energy frame contains the spatial gradient of the thermal potential $\alpha=(\rho+p)/nT-s$, and not the temperature gradient or acceleration, which arise in the particle frame. We can complete the comparison by using the Gibbs equation (4) and the momentum conservation equation (A2) to show that

$$\left(\frac{nT}{\rho+p}\right)\mathrm{D}_a\alpha=-\frac{1}{T}\mathrm{D}_aT-\dot{u}_a+\frac{1}{\rho+p}\left(\mathrm{D}_aB+\mathrm{D}^b\pi_{ab}\right) \tag{B5}$$

in the energy frame. It follows that the energy-frame equation (2.38b) of [4] is in fact of the same form as the particle-frame equation (2.41b).

In the cosmological setting, all the terms representing coupling of effects are nonlinear and may be neglected. To first order, $\dot{Q}_{\langle a\rangle}=\dot{Q}_a$ and $\dot{\pi}_{\langle ab\rangle}=\dot{\pi}_{ab}$. Furthermore, we follow the usual practice of neglecting the final term in square brackets on the right hand side of each transport equation. This can lead to problematic behavior in some cases, as shown in [12,13] for the case of bulk viscosity. However, under a range of reasonable conditions, these terms may be neglected in comparison with the remaining terms [14,21]. With these assumptions, the terms in braces in the transport equations (B1)–(B3) all fall away, leading to the simplified cosmological transport equations (40)–(42).

[9]Note that the a_1 in equation (7.1c) of [4] should be α_1.

[10]Note that ∇_aq^a in equation (21) of [9] should be $q^{ab}\nabla_aq_b$ (using their notation).

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[18] We use units with $8\pi G = 1 = c$ and $k_{\rm B} = 1$; $a, b, \cdots$ are spacetime indices; (square) round brackets enclosing indices denote (anti-) symmetrization, and angle brackets denote the spatially projected, symmetric and trace-free part. The main differences between our notation and that of [5] are given by: $\mathrm{D}_a \equiv {}^{(3)}\nabla_a$, $\rho \equiv \mu$, $\delta_a \equiv \mathcal{D}_a$, $\delta \equiv \Delta$, $\theta_a \equiv \mathcal{Z}_a$, $p_a \equiv wP_a$, $e_a \equiv wnT\mathcal{E}_a/r$, $\xi_{ab} \equiv \Sigma_{ab}$.
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